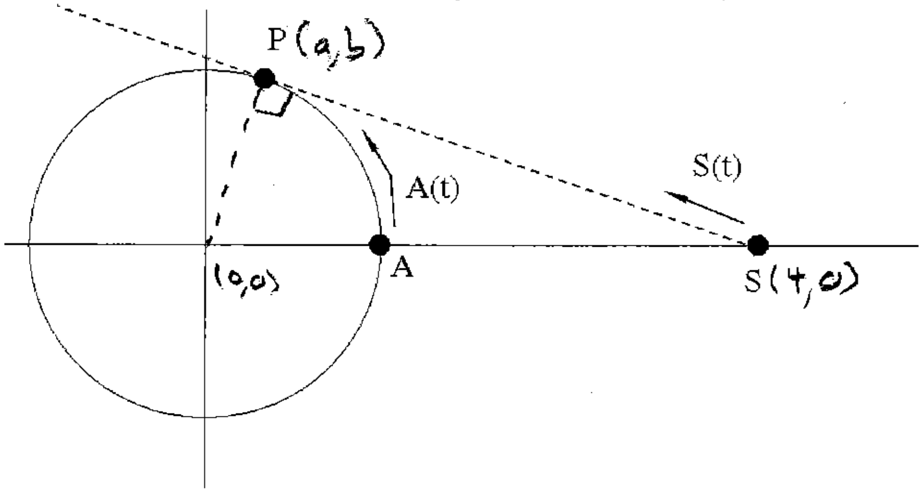
Closing Mon: 10.1

Closing Fri: 2.1, 2.2, 2.3

Warning: Expect a lot of work.

Visit the MSC! *Check newsletter hints.*

Entry Task (directly from HW):



An ant is walking around the unit circle such that: $x = \cos(\pi t), y = \sin(\pi t)$.

Starting at the same time, a spider walks from (4,0) along a line tangent to the circle, as shown.

1. Find the point P.
2. When will the ant first get to this point? Second Time?
3. Give the parametric linear equations for the spider in order for it to get the point P at ~~the same time~~ this second time

$\boxed{1}$ LABEL P(a,b)

FACT 1: (a,b) IS ON THE UNIT CIRCLE

SO $a^2+b^2=1$

FACT 2: slope of tangent $= \frac{b-0}{a-4}$

FACT 3: ALSO,
slope of tangent $= -\frac{1}{\left(\frac{b-0}{a-0}\right)} = -\frac{a}{b}$

SO (i) $a^2+b^2=1$ AND

(ii) $\frac{b}{a-4} = -\frac{a}{b} \Rightarrow b^2 = -a^2+4a$
$\Rightarrow a^2+b^2=4a$

COMBINING CONDITIONS

(i) AND (ii) YIELDS $1 = 4a \Rightarrow \boxed{a = 1/4}$

AND $a^2+b^2=1 \Rightarrow b = \pm\sqrt{1-a^2}$

$\Rightarrow b = \sqrt{1-(1/4)^2} = \sqrt{\frac{15}{16}}$
$= \sqrt{15}/4$

$$\boxed{P = \left(\frac{1}{4}, \frac{\sqrt{15}}{4}\right)}$$

2 ANT'S LOCATION IS GIVEN BY

$$x = \cos(\pi t) \stackrel{?}{=} 1/4$$
$$y = \sin(\pi t) \stackrel{?}{=} \sqrt{15}/4$$

Solving gives

$$\pi t = \cos^{-1}(1/4)$$
$$\Rightarrow t = \frac{1}{\pi}\cos^{-1}(1/4)$$
$$\boxed{t \approx 0.4195694 \text{ seconds}}$$

ASIDE: SINCE IT TAKES 2 SECONDS TO DO A FULL ROTATION, THE SECOND TIME WILL BE 2 SECONDS LATER!

$$t \approx 2.4195694 \text{ seconds}$$

(second time)

3 WANT TO FIND THESE

$$x = \boxed{x_0} + \boxed{v_x} t$$
$$y = \boxed{y_0} + \boxed{v_y} t$$

SUCH THAT

WHEN $t=0$, $x=4$ and $y=0$

AND

WHEN $t = 2.4195694$, $x = 1/4$ and $y = \sqrt{15}/4$

THUS, $x_0 = 4$, $y_0 = 0$

$$v_x = \frac{x - x_0}{t} = \frac{1/4 - 4}{2.4195694}$$
$$\approx -1.54986256$$

$$v_y = \frac{y - y_0}{t} = \frac{\sqrt{15}/4 - 0}{2.4195694}$$
$$\approx 0.40017279$$

$$\boxed{\begin{aligned} x &= 4 - 1.54986256\, t \\ y &= 0 + 0.40017279\, t \end{aligned}}$$

Linear Motion: $x = x_0 + v_x t$

$y = y_0 + v_y t$

$(x_0, y_0) =$ initial location

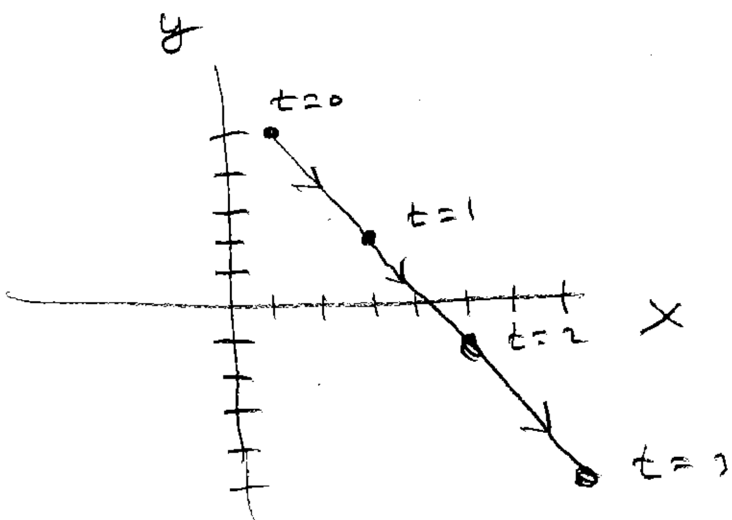
$v_x =$ horizontal velocity $= \frac{\Delta x}{\Delta t}$

$v_y =$ vertical velocity $= \frac{\Delta y}{\Delta t}$

Example: $x = 1 + 2t$

$y = 5 - 3t$

t	0	1	2	3
x	1	3	5	7
y	5	2	-1	-4



Circular Motion: $x = r\cos(\theta_0 + \omega t) + x_c$

$y = r\sin(\theta_0 + \omega t) + y_c$

$(x_c, y_c) =$ center of circle

r = radius of circle

θ_0 = initial angle

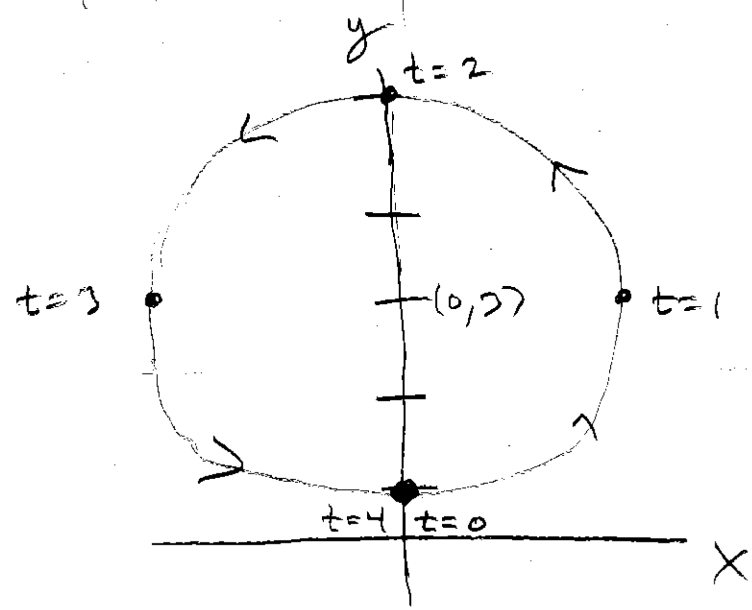
ω = angular speed = $\Delta\theta/\Delta t$

Example: $x = 2\cos\left(\frac{3\pi}{2} + \frac{\pi}{2}t\right)$

$y = 3 + 2\sin\left(\frac{3\pi}{2} + \frac{\pi}{2}t\right)$

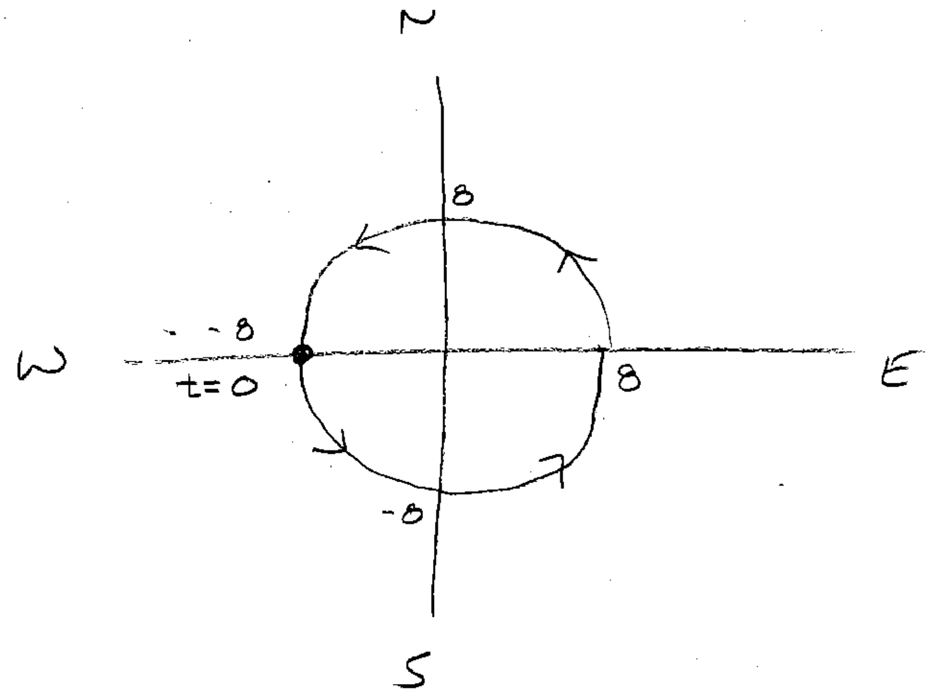
$(x_c, y_c) = (0, 3)$ $r = 2$

$\theta_0 = 3\pi/2$, $\omega = \pi/2 \ \frac{rad}{sec}$



Example:

A bug follows a circular path with radius 8 inches. It starts at the west-most edge. It rotates counterclockwise at a constant 10 revolutions per minute.



Give the equations for motion in terms of time *t*.

r = ??

θ_0 = ?? (give in radians)

w = ?? (give in radians/min)

$r = 8$ $(x_c, y_c) = (0,0)$

$\theta_0 = \pi$

$$\omega = 10 \frac{\text{rev}}{\text{min}} \cdot \frac{2\pi \text{ rad}}{1 \text{ rev}} = 20\pi \frac{\text{rad}}{\text{min}}$$

$$\boxed{\begin{aligned} x &= 8\cos(\pi + 20\pi t) \\ y &= 8\sin(\pi + 20\pi t) \end{aligned}}$$

Overview of Trigonometric Functions Values and Basic Facts

If r is the radius of a circle and θ is an angle measured from standard position, then we can find the corresponding location on the edge of the circle by using the formulas

$$x = r\cos(\theta) = r\cos(\theta_0 \pm wt) \text{ and } y = r\sin(\theta) = r\sin(\theta_0 \pm wt)$$

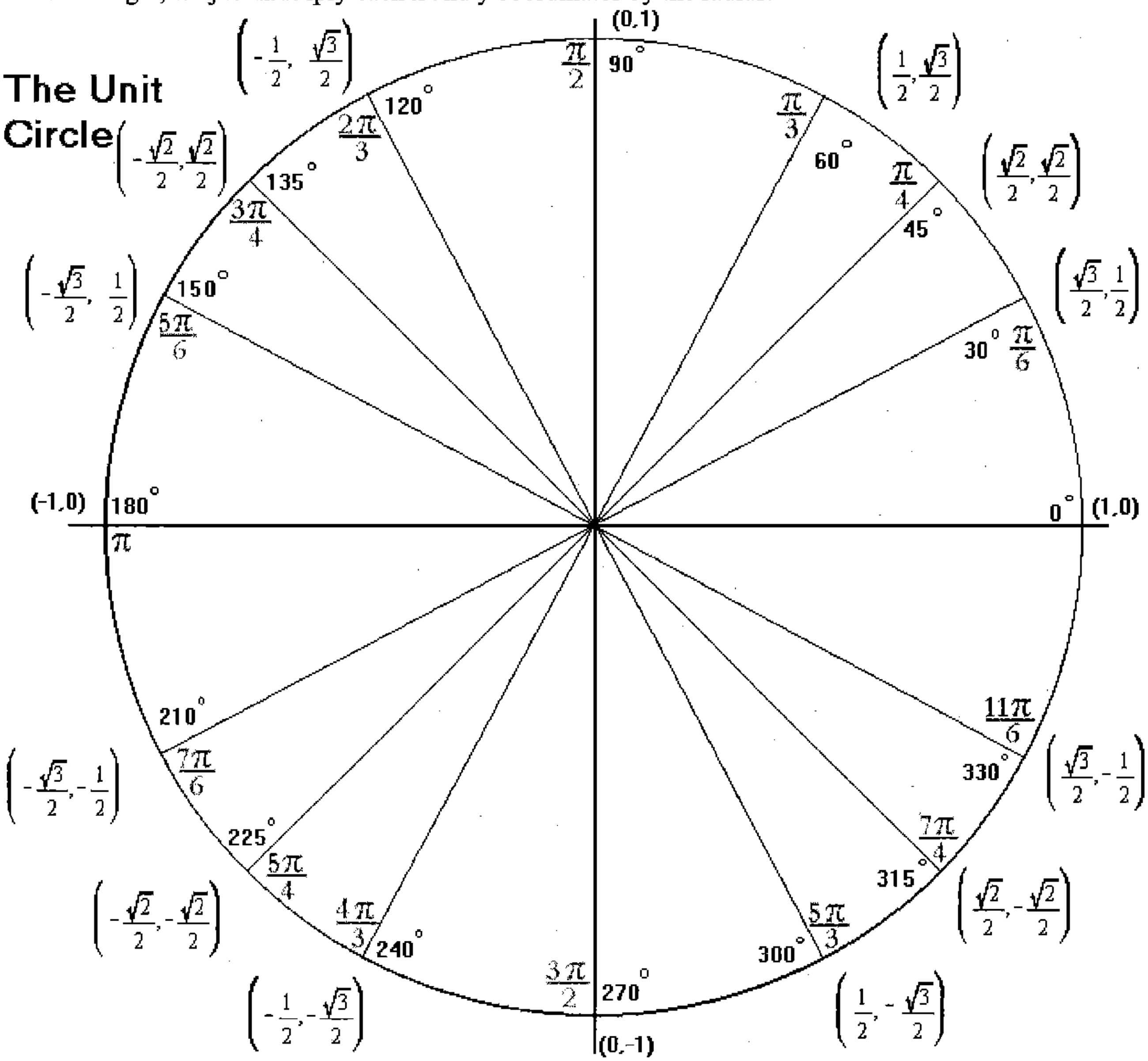
For most values of θ, $\sin(\theta)$ and $\cos(\theta)$ are not easily computed and require a calculator. However, you are expected to know the following values:

Angle		$\sin(\theta)$	$\cos(\theta)$
0 deg	0 rad	0	1
30 deg	π/6 rad	1/2	$\sqrt{3}/2$
45 deg	π/4 rad	$\sqrt{2}/2$	$\sqrt{2}/2$
60 deg	π/3 rad	$\sqrt{3}/2$	1/2
90 deg	π/2 rad	1	0

You can find the other trig function values at these angles using the relationships:

$\tan(\theta) = \sin(\theta)/\cos(\theta)$, $\cot(\theta) = \cos(\theta)/\sin(\theta)$, $\csc(\theta) = 1/\sin(\theta)$, $\sec(\theta) = 1/\cos(\theta)$.

Often these values are remembered by actually putting them on a circle. Here is the circle with radius 1 (or the *unit circle*) with the values at the above angles label along with corresponding angles in other quadrants. If the radius is larger, we just multiply each x and y coordinates by the radius.



Ch. 2 Limits and Derivatives

2.1 Motivation

Calculus is primarily about "rates".

$$\text{rate} = \frac{\text{change in quantity}}{\text{change in time}}$$

We will find *instantaneous* rates, by building a limiting process of better and better approximations.

Example: The distance traveled by an object is recorded at various times:

t (seconds)	0	1	2	3
Dist (meters)	0	1.2	4.5	10.4

1. What is the average velocity
   ... from t = 0 to t = 3?
   ... from t = 1 to t = 3?
   ... from t = 2 to t = 3?
2. What is the instantaneous velocity at t = 3?

$\boxed{0 \text{ to } 3}$ $\quad \frac{\Delta \text{DIST}}{\Delta \text{TIME}} = \frac{10.4-0}{3-0} = 3.4\overline{6}$ m/s

$\boxed{1 \text{ to } 3}$ $\quad \frac{\Delta \text{DIST}}{\Delta \text{TIME}} = \frac{10.4-1.2}{3-1} = 4.6$ m/s

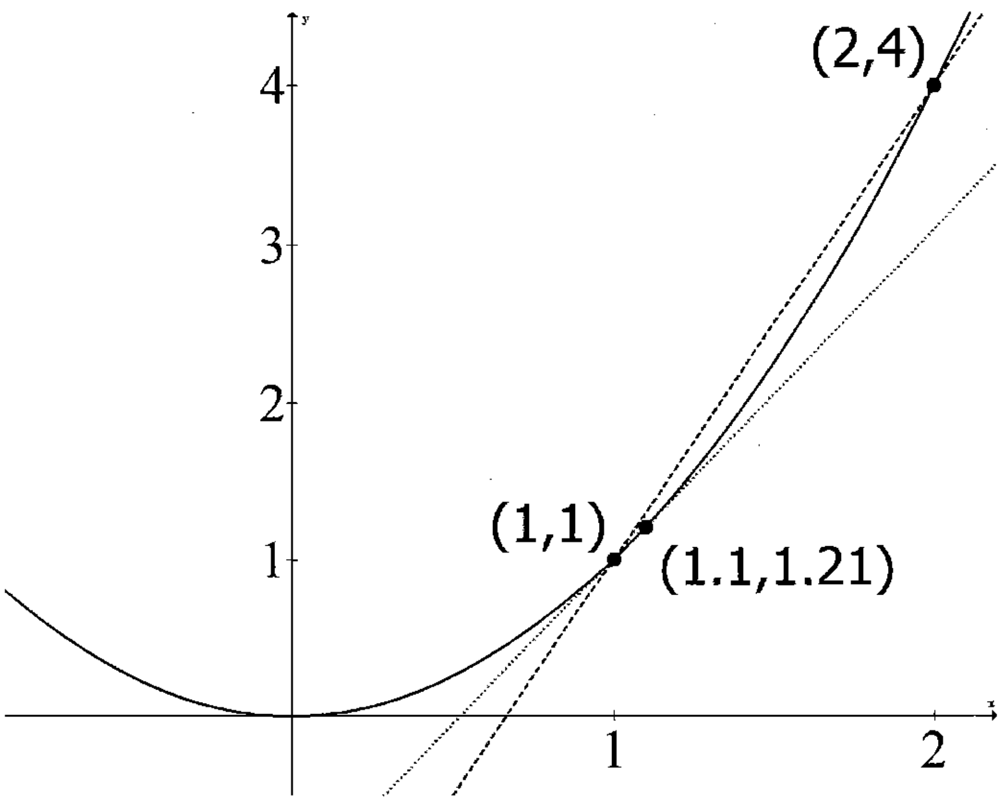
$\boxed{2 \text{ to } 3}$ $\quad \frac{\Delta \text{DIST}}{\Delta \text{TIME}} = \frac{10.4-4.5}{3-2} = 5.9$ m/s

WE DON'T KNOW INS. VELOCITY AT 3
BUT WE MIGHT SUSPECT
THAT THE VELOCITY FROM 2 TO 3
IS THE BEST APPROXIMATION.

Example:

Consider the function: $f(x) = x^2$

1. Find the slope of the *secant* line from x = 1 to x = 2.
2. Find the slope of the secant line from x = 1 to x = 1.1.



$\boxed{1}$ $\frac{\Delta y}{\Delta x} = \frac{f(2) - f(1)}{2 - 1} = \frac{(2)^2 - (1)^2}{2 - 1}$

$= \frac{4 - 1}{2 - 1} = \frac{3}{1} = 3$

$\boxed{2}$ $\frac{\Delta y}{\Delta x} = \frac{f(1.1) - f(1)}{1.1 - 1} = \frac{(1.1)^2 - (1)^2}{1.1 - 1}$

$= \frac{1.21 - 1}{0.1} = \frac{0.21}{0.1}$

$= 2.1$

SAME AS

$\frac{f(1 + 0.1) - f(1)}{1 + 0.1 - 1}$

In this course we will find

f'(1) = `slope of the tangent at *x=1*'

$$= \lim_{h \to 0} \frac{f(1 + h) - f(1)}{h}$$